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Normalization sum rule and spontaneous breaking of U(N) invariance in random matrix ensembles

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It is shown that the two-level correlation function $R_2^\infty(s,s')$ in the invariant random matrix ensembles (RME's) with soft confinement exhibits unexpected long-range correlations in the form of a sharp peak at $s\approx -s'$. The presence of such a "ghost" correlation peak lifts the sum rule prohibition for the level number variance to have a Poisson-like term $\mathrm{var}(n)=\eta n$ that is typical of RME's with broken U(N) symmetry. Thus we conclude that the U(N) invariance is broken spontaneously in the RME's with soft confinement, η playing the role of an order parameter.

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The statistical description of complex systems by ensembles of random matrices turned out to be a powerful general approach that was successively applied to a great variety of systems in different fields from nuclear physics [1] to mesoscopics [2] and quantum chaos [3].

The classical random matrix theory (RMT) by Wigner, Dyson and Mehta [1] describes the statistics of eigenvalues for a Gaussian ensemble of random Hermitian $N\times N$ matrices $\mathbf{H}$ with the probability distribution $P(\mathbf{H}) \propto \exp[-\mathrm{Tr}\mathbf{H}^2]$. By definition, the statistical properties of this ensemble are invariant under unitary transformations U(N) of matrices $\mathbf{H}$ and thus there is no basis preference in the RMT. This means that the classical RMT can be applied only to quantum systems where all (normalized) linear combinations of eigenfunctions have similar properties.

For disordered electronic systems, it implies that all eigenstates must be extended. In other words, the classical RMT is applicable only for describing the energy level statistics in the metal phase [4,5] that exists in the dimensionality $d>2$ at a relatively weak disorder.

With disorder increasing the system goes through the Anderson transition to an insulating phase in which all eigenstates are localized. The level statistics in this phase obviously cannot be described by the U(N)-invariant random matrix ensemble (RME), since one can construct an extended state by a linear combination of localized states randomly positioned throughout the sample. Thus the proper probability distribution $P(\mathbf{H})$ must contain a basis preference in order to exclude unitary transformations which would lead to formation of such extended states.

The ensemble of random banded matrices (RBME) [6,7] is an example of such a noninvariant RMT. It describes properties of systems belonging to so-called quasi-one-dimensional universality class which includes quasi-one-dimensional disordered electronic systems with localization [6] and certain quantum chaotic systems [7]. The corresponding eigenvalue statistics are Poissonian in the $N\to\infty$ limit (at a fixed bandwidth b) and reduce to the Wigner-Dyson form at $b\to\infty$. Thus changing the parameter b/N, one can describe the crossover from Wigner-Dyson to Poisson level statistics which occurs in quasi-one-dimensional disordered systems with increasing the ratio L/ξ of the sample length L and the localization radius ξ.

While the localization in quasi-one-dimensional systems seems to be well described in terms of the RBME, the problem of the random matrix description of the critical region near the Anderson transition and the Anderson insulator phase for $d>2$ remains open. The recent works [8–10] where the existence of the universal critical level statistics has been demonstrated resulted in an intensive search for the proper RME description.

In this connection, two different generalizations of the classical RMT have been recently proposed [11,12]. The generalized RME studied in Ref. [11] was obtained from the Gaussian invariant ensemble by introducing a symmetry-breaking term:

$$P(\mathbf{H})\propto e^{-\mathrm{Tr}\mathbf{H}^2}\, e^{-h^2N^2\mathrm{Tr}([\mathbf{\Lambda},\mathbf{H}][\mathbf{\Lambda},\mathbf{H}]^\dagger)}. \quad (1)$$

The h-dependent term breaks the U(N) invariance and tends to align $\mathbf{H}$ with a symmetry-breaking unitary matrix $\mathbf{\Lambda}$ thus setting the basis preference. It turned out [11] that even after averaging over $\mathbf{\Lambda}$ the resulting ensemble leads to the eigenvalue statistics that deviate from the Wigner-Dyson form. The difference between the Wigner-Dyson statistics that correspond to $h=0$ and the level statistics for *any nonzero h* turns out to be dramatic in the thermodynamic (TD) limit $N\to\infty$. Namely, for $h\neq 0$ the variance var(n) of the number of levels in an energy window that contains n levels on the average grows linearly with n at $n\gg 1$:

$$\mathrm{var}(n)=\langle(\delta n)^2\rangle=\eta(h)\ n\sim h\ n,\quad \eta(0)=0. \quad (2)$$

For the classical RMT [1], $\mathrm{var}(n)\propto \ln n$, which is negligible as compared to Eq. (2) for *any* nonzero $0<\eta(h)<1$ in the limit $n\to\infty$. The Poisson-like behavior described by Eq. (2) is valid also for an RBME in the TD limit.

In contrast to Eq. (1), the probability distribution suggested in Ref. [12] is explicitly U(N) invariant:

$$P(\mathbf{H})\propto e^{-\mathrm{Tr}V(\mathbf{H})}. \quad (3)$$

The only singularity in this model is that the (even in E) "confining potential" $V(E)$ grows extremely slowly:

$$V(E)=\tfrac{A}{2}\ln^2|E|,\quad |E|\to\infty. \quad (4)$$

However, the Poisson-like behavior, Eq. (2), turns out to be valid for this model too, provided that the energy window does not contain the origin $E=0$.

It should be stressed [13] that for steeper confining potentials, $V(E)=|E|^{\alpha}$, no deviation from the Wigner-Dyson statistics was observed in the bulk of the spectrum, so that $\eta=0$ for *all* $\alpha>0$. What happens with the invariant RME at the transition from a power law to logarithmic confinement looks like the spontaneous breaking of the U(N) symmetry.

In this paper we present both analytical and numerical arguments showing this new phenomenon to exist. We show the parameter η to play the role of an order parameter and clarify its connection with the breaking of the normalization sum rule in the TD limit.

The reason why $\eta=0$ for a wide class of invariant RME's is connected with the normalization sum rule:

$$\int_{-\infty}^{+\infty} Y_2^{(N)}(s,s')ds'=1. \tag{5}$$

The cluster function $Y_2^{(N)}(s,s')=\delta(s-s')-R_2^{(N)}(s,s')$ is related to the two-level correlation function $R_2^{(N)}(s,s')$:

$$R_2^{(N)}(s,s')=\frac{\langle\rho(E_s)\rho(E_{s'})\rangle}{\langle\rho(E_s)\rangle\,\langle\rho(E_{s'})\rangle}-1. \tag{6}$$

where $\rho(E)=\mathrm{Tr}\{\delta(E-\mathbf{H})\}$ is the level density, $\langle\cdots\rangle$ denotes ensemble averaging with the probability distribution $P(\mathbf{H})$, and the new variable $s(E)$ is chosen so that the mean level density in this variable $\langle\tilde{\rho}(s)\rangle=\langle\rho(E_s)\rangle dE_s/ds=1$ for $-N/2<s<N/2$.

For N finite, Eq. (5) is an exact property of any RME that follows simply from the normalization condition $\int\rho(E)dE=N$. However, the sum rule may be violated [14] in the limit $N\rightarrow\infty$, since after the integration in Eq. (5) both terms in Eq. (6) result in the divergent constants N being subtracted from each other. Below we will assume this limit to be taken.

The general relationship between the cluster function $Y_2^{\infty}(s,s')$ and the coefficient $\eta=\lim_{n\rightarrow\infty}\{d[\mathrm{var}(n)]/dn\}$ in the level number variance, Eq. (2), for an energy window centered at a point $E_0=E(s_0)$, reads

$$\eta=\lim_{n\rightarrow\infty}\left[1-\int_{a_-}^{a_+}[Y_2^{\infty}(a_+,s')+Y_2^{\infty}(a_-,s')]\,ds'\right], \tag{7}$$

where $a_{\pm}=s_0\pm n/2$. In the case where the cluster function in the TD limit $Y_2^{\infty}(s,s')=Y_2^{\infty}(s-s')$ is translationally invariant, the relationship reduces to

$$\eta=1-\int_{-\infty}^{+\infty}Y_2^{\infty}(s)\,ds. \tag{8}$$

Now comparing Eq. (5) and Eq. (8) we can conclude that the parameter η is equal to the deficiency of the sum rule in the $N\rightarrow\infty$ limit [9].

Whether the sum rule is broken or not depends on the behavior of the effective level interaction at large distances. Quite generally, the joint probability distribution $P[\{x_i\}]$ of eigenvalues x_i of a matrix $\mathbf{H}$ can be represented in the form similar to the Gibbs distribution $P[\{x_i\}]=\exp[-\beta\mathscr{H}]$ for the one-dimensional plasma of classical particles described by the potential energy functional $\mathscr{H}[\{x_i\}]=\Sigma_i V(x_i)+W[\{x_i\}]$. For the unitary ensembles considered in this paper, the effective temperature $\beta=2$. In case of the invariant ensembles given by Eq. (3), the many-body interaction $W[\{x_i\}]$ reduces to the pairwise logarithmic effective level interaction [1]:

$$W[\{x_i\}]=-\sum_{i>j}\ln|x_i-x_j|. \tag{9}$$

For an RME similar to that given by Eq. (1), the corresponding effective level interaction that appears after averaging [11] over the symmetry-breaking matrix $\mathbf{\Lambda}$ contains all the many-body terms and can be rewritten in the form of a determinant Det_{ij}, $i,j=1,\ldots,N$:

$$W[\{x_i\}]=-\tfrac{1}{2}\ln\,\mathrm{Det}_{ij}[e^{-h^2N^2(x_i-x_j)^2}]. \tag{10}$$

In both limiting cases, $|x_i-x_j|\rightarrow 0$ and $|x_i-x_j|\rightarrow\infty$, the determinant in Eq. (10) can be approximated by a pairwise interaction of the form

$$W[\{x_i\}]=-\tfrac{1}{2}\sum_{i>j}\ln(1-e^{-2h^2N^2(x_i-x_j)^2})+\mathrm{const.} \tag{11}$$

The main difference between Eq. (9) and Eq. (11) is that the symmetry-breaking field $h\neq 0$ results in the cutoff of the effective level interaction at sufficiently large distances [15]. This difference is crucial for the fulfillment of the sum rule, Eq. (5), in the TD limit.

It can be shown that the normalization sum rule persists also in the TD limit if the effective level interaction is long range (nonintegrable). We will present here a simple proof of this statement based on the mean-field approach [16] that is valid exactly for the long-range interactions in the $N\rightarrow\infty$ limit. Suppose that the effective level interaction $f(x_i-x_j)$ is pairwise and long range. Then using the relationship $\delta\langle\rho(E)\rangle/\delta V(E')=-\beta[\langle\rho(E)\rho(E')\rangle-\langle\rho(E)\rangle\,\langle\rho(E')\rangle]$ and the Dyson mean-field equation [1] for $\langle\rho(E)\rangle$

$$\int_{-\infty}^{+\infty}\langle\rho(E'')\rangle f(E''-E')+V(E')=\mathrm{const}, \tag{12}$$

one can derive the integral equation for the two-level correlation function [16]:

$$\int_{-\infty}^{+\infty}ds''R_2^{\infty}(s,s'')\,f(E_{s''}-E_{s'})=\beta^{-1}\delta(s-s'). \tag{13}$$

Now integrate this equation over all s changing the order of integration in the left-hand side and denote the integral $\int_{-\infty}^{+\infty}R_2^{\infty}(s,s'')ds=I$:

$$I\beta\int_{-\infty}^{+\infty}f(E_{s''}-E_s)\,ds''=1. \tag{14}$$

From the definition of the function E_s just after Eq. (6) and a physically obvious property $d\langle\rho(E)\rangle/d|E|<0$ that holds for an even confinement potential $V(E)$, it follows that E_s increases linearly or faster with s. Therefore for any long-range interaction with $\int f(E)dE$ divergent at infinity, the integral in Eq. (14) is also divergent. Thus we arrive at the relationship

$$I = 1 - \int_{-\infty}^{+\infty} Y_2^{\infty}(s,s')ds' = 0, \qquad (15)$$

which proves the validity of Eq. (5) in the limit $N \to \infty$. On the contrary, if the effective interaction $f(E-E')$ is cut at large distances, the quantity I in Eqs. (14) and (15) must be finite and thus the sum rule is violated.

We see that the symmetry-breaking term in Eq. (1) leads to the cutoff of the effective level interaction at large distances and thus to the breakdown of the normalization sum rule in the TD limit. This results in a quasi-Poisson level number variance, Eq. (2), governed by the nonzero parameter η, Eq. (8).

The situation resembles the appearance of the long-range order in spin systems in the external magnetic field $\mathbf{h}$ which breaks down the rotational invariance. In this case, the spin-spin correlator $\langle \mathbf{S}(\mathbf{r})\mathbf{S}(\mathbf{r}')\rangle \to m^2$ is constant at large distances, the magnetization $|m|$ depending linearly on $|\mathbf{h}|$. Since the spin-spin correlator is invariant under global rotations of the spins, the order parameter, that is $|m|$, is unchanged after averaging over the direction of the symmetry-breaking field $\mathbf{h}$.

In our problem, the quantity analogous to $\langle \mathbf{S}(\mathbf{r})\mathbf{S}(\mathbf{r}')\rangle$, which is invariant under global (independent of E, E') unitary transformations, is the two-level correlation function $\langle \mathrm{Tr}\{\delta(E-\mathbf{H})\}\mathrm{Tr}\{\delta(E'-\mathbf{H})\}\rangle$. The symmetry-breaking field $h\mathbf{\Lambda}$ plays the role of the magnetic field, and the parameter η is similar to the magnetization. Like $|m|$, η remains nonzero after averaging over $\mathbf{\Lambda}$.

Now we return to consider the TD limit of the invariant ensemble with soft confinement, defined by Eqs. (3) and (4). Since the effective level interaction, Eq. (9), is long range, the sum rule, Eq. (5), must hold for any confining potential $V(E)$. However, for soft confinement this does not necessarily lead to $\eta=0$. The reason is the dramatic breakdown of the translational invariance of the cluster function $Y_2^{\infty}(s,s') = Y_2^{\infty}(s-s') + Y_a(s,s')$ which turns out to have an anomalous part $Y_a(s,s')$ that exhibits a sharp peak of width A near $s=-s'$.

With the translational invariance broken in such a way, the level number variance var(n) becomes dependent on the position of the energy window s_0, and can oscillate as a function of n. The parameter η is given by the general Eq. (7), where $n=1,2,\ldots$. It is easy to show that the integral in Eq. (7) reduces approximately to that in the sum rule, Eq. (5), only provided that $n/2 - |s_0| \gg A$, that is, the origin $E=s=0$ is far within the energy window. This means that the parameter η vanishes for a symmetric window $s_0=0$ and it is nonzero if $|s_0| \gg n/2$.

Indeed, the Monte Carlo simulations on the classical one-dimensional plasma described by Eqs. (4) and (9) show a dramatic difference in the level number variance in these two cases. A remarkable property of the model is that, for a symmetric window, the level number variance var(n) is *constant* for all integers $n \gg 1$. Thus the "level rigidity" is even higher than that for the classical RMT where var$(n) \propto \ln n$.

We will now show the existence and importance of the anomalous part $Y_a(s,s')$ through a simplified derivation of the cluster function $Y_2^{\infty}(s,s')$ for small values of the parameter $q = e^{-\pi^2 A}$. Though $Y_a(s,s')$ can be extracted from the exact solution for the confining potential obeying Eq. (4), its existence was not mentioned in Ref. [12].

We start with the representation [1] of $Y_2^{\infty}(s,s')$ in terms of the orthonormal "wave functions" $\varphi_i(E) = p_i(E)\exp[-V(E)]$, where $p_i(E)$ are orthogonal polynomials corresponding to the weight function $e^{-2V(E)}$:

$$Y_2^{\infty}(s,s') = \frac{K^2(E_s,E_{s'})}{K(E_s,E_s)K(E_{s'},E_{s'})}. \qquad (16)$$

Here the kernel $K(E,E')$ is given by [1]

$$K(E,E') = \frac{1}{\pi C^2}\,\frac{\varphi_o(E')\varphi_e(E) - \varphi_e(E')\varphi_o(E)}{E'-E}, \qquad (17)$$

where C is a constant and $\varphi_{o(e)}$ are $N \to \infty$ limits of wave functions $\varphi_N(E)$ of the odd and even order, respectively.

For $q \ll 1$, the wave functions $\varphi_{o(e)}(E)$ have a "semiclassical" form that is a generalization of the $N \to \infty$ limit of the oscillator wave functions for a nonlinear $s(E)$:

$$\varphi_o(E) = C\sin[\pi s(E)], \quad \varphi_e(E) = C\cos[\pi s(E)]. \qquad (18)$$

The arguments in sin and cos are chosen so that the density of zeros $\rho_0 = ds/dE$ coincides with the mean level density $\langle \rho(E)\rangle$.

The function $s(E)$ can be easily found from the known solution of the Dyson mean-field equation (12). Solving Eq. (12) with $f(E-E') = -\ln|E-E'|$ for the confining potential, Eq. (4), and taking the limit $N \to \infty$, we have

$$\langle \rho(E)\rangle = ds/dE = \tfrac{A}{2|E|}, \quad E_s = \lambda e^{2|s|/A}\,\mathrm{sgn}(s), \qquad (19)$$

where λ is a constant of integration.

Now using Eqs. (16)–(19) and the relationship [1] $\langle \rho(E)\rangle = K(E,E)$ one can obtain an explicit form of the cluster function $Y_2(s,s')$ for $q = e^{-\pi^2 A} \ll 1$, where A is a coefficient in Eq. (4). For $ss'>0$ we arrive at the translationally invariant expression [12]:

$$Y_2^{\infty}(s-s') = \frac{1}{\pi^2 A^2}\,\frac{\sin^2[\pi(s-s')]}{\sinh^2[(s-s')/A]}, \quad ss'>0. \qquad (20)$$

However, there are strong correlations for $ss'<0$ too:

$$Y_a(s,s') = \frac{1}{\pi^2 A^2}\,\frac{\sin^2[\pi(s-s')]}{\cosh^2[(s+s')/A]}, \quad ss'<0. \qquad (21)$$

This is just the anomalous part of the cluster function discussed above. Its remarkable property is a sharp peak at $s \approx -s'$ with a height that *does not* decrease when $|s-s'| \sim 2|s| \to \infty$.

The existence of such a "ghost" peak in the two-level correlation function is confirmed also by the Monte Carlo simulations on the effective plasma model with a soft confinement (see Fig. 1).

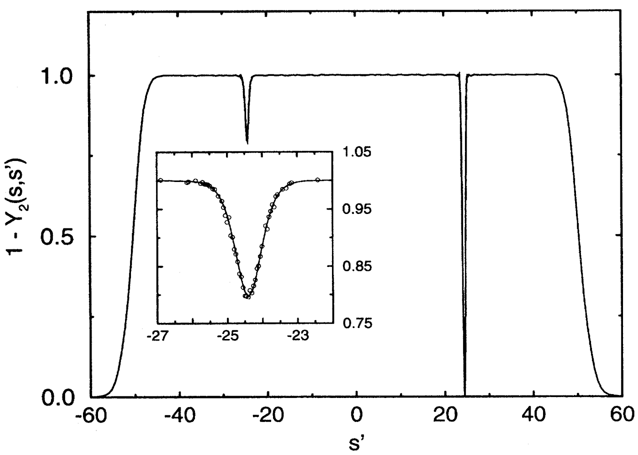

FIG. 1. Monte Carlo results showing the existence of the "ghost" correlation peak. The simulations are performed for $\beta=2$ with $N=100$ "particles" in the confining potential $V(E)=\frac{A}{2}\ln^2(1+|E|)$ for $A=0.5$. The reference particle is mobile around $s=24$. The solid line in the inset corresponds to Eq. (21) with oscillations being averaged out.

Such anomalous, infinitely long-range level correlations make it possible for the *invariant* random matrix ensemble defined by Eqs. (3) and (4) to exhibit the Poisson-like level number variance $\mathrm{var}(n)=\eta n$ which is typical for an RME with *broken* U(N) symmetry. Moreover, after the substitution $\pi^2 A=2/h \gg 1$, the form of the "normal," translationally invariant part of $Y_2^{\infty}(s,s')$ given by Eq. (20) is identical to that found in Ref. [11] for the matrix ensemble, Eq. (1), with symmetry breaking.

This is a crucial point in the chain of arguments in favor of the statement that the U(N) symmetry is *spontaneously broken* in the random matrix ensemble, Eqs. (3) and (4), with soft confinement.

The corresponding order parameter which arises either due to the external symmetry-breaking field h or spontaneously for a sufficiently soft confinement, can be defined in the limit $N\rightarrow\infty$ as follows:

$$\eta=\lim_{s\rightarrow+\infty}\left[1-\int_{s'>0}Y_2^{\infty}(s,s')\,ds'\right]. \tag{22}$$

It coincides with the coefficient in the quasi-Poisson term in the level number variance, Eq. (2), provided that the energy window does not contain the origin.

In the case of the spontaneously broken U(N) symmetry, a complementary order parameter can be introduced:

$$\tilde{\eta}=\lim_{s\rightarrow+\infty}\left[-\int_{s'<0}Y_2^{\infty}(s,s')\,ds'\right], \tag{23}$$

which involves integration only over the anomalous part of the cluster function. Because of the sum rule, Eq. (5), the sum $\eta+\tilde{\eta}=0$ must be zero. Exploiting the spin analogy discussed above, one can say that the spontaneous symmetry breaking is of the antiferromagnetic type, with η and $\tilde{\eta}$ playing a role of the sublattice magnetizations.

We conclude that the anomalous statistics of eigenvalues in RME's with soft confinement is a result of the spontaneous breaking of the U(N) symmetry which exhibits itself in the quasi-Poisson term in the level number variance and in the appearance of the "ghost" peak at $s=-s'$ in the two-level correlation function $R_2^{\infty}(s,s')$.

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[1] See, for reviews, M. L. Mehta, *Random Matrices* (Academic Press, Boston, 1991); F. Haake, *Quantum Signatures of Chaos* (Springer-Verlag, Berlin, 1991).

[2] B. L. Altshuler, V. E. Kravtsov, and I. V. Lerner, in *Mesoscopic Phenomena in Solids*, edited by B. L. Altshuler, P. A. Lee, and R. A. Webb (North-Holland, Amsterdam, 1991), p. 449.

[3] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, New York, 1990).

[4] L. P. Gor'kov and G. M. Eliashberg, Zh. Eksp. Teor. Fiz. **48**, 1407 (1965) [Sov. Phys. JETP **21**, 940 (1965)].

[5] K. B. Efetov, Adv. Phys. **32**, 53 (1983).

[6] Y. V. Fyodorov and A. D. Mirlin, Phys. Rev. Lett. **69**, 1093 (1992); **71**, 412 (1993).

[7] F. M. Izrailev, Phys. Rep. **196**, 299 (1990); G. Casati, L. Molinari, and F. M. Izrailev, Phys. Rev. Lett. **64**, 1851 (1990).

[8] B. I. Shklovskii, B. Shapiro, B. R. Sears, P. Lambrianides, and H. B. Shore, Phys. Rev. B **47**, 11 487 (1993).

[9] V. E. Kravtsov, I. V. Lerner, B. L. Altshuler, and A. G. Aronov, Phys. Rev. Lett. **72**, 888 (1994).

[10] A. G. Aronov, V. E. Kravtsov, and I. V. Lerner, Phys. Rev. Lett. **74**, 1174 (1995).

[11] J.-L. Pichard and B. Shapiro, J. Phys. **4**, 623 (1994); M. Moshe, H. Neuberger, and B. Shapiro, Phys. Rev. Lett. **73**, 1497 (1994).

[12] K. A. Muttalib, Y. Chen, M. E. H. Ismail, and V. N. Nicopoulos, Phys. Rev. Lett. **71**, 471 (1993); C. Blecken, Y. Chen, and K. A. Muttalib, J. Phys. A **27**, L563 (1994).

[13] C. M. Canali, Mats Wallin, and V. E. Kravtsov, Phys. Rev. B **51**, 2831 (1995).

[14] V. E. Kravtsov and I. V. Lerner, Phys. Rev. Lett. **74**, 2563 (1995); A. G. Aronov and A. D. Mirlin, Phys. Rev. B **51**, 6131 (1995).

[15] Distances $\Delta x\sim 1/hN$ correspond to $\Delta s\sim h^{-1/2}$.

[16] C. W. J. Beenakker, Phys. Rev. Lett. **70**, 1155 (1993); R. A. Jalabert, J.-L. Pichard, and C. W. J. Beenakker, Europhys. Lett. **24**, 1 (1993).